

Is a color superconductor topological ?

Phys. Rev. D 81, 074004 (2010) [arXiv:1001.2555]

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“P- and CP-odd Effects in Hot and Dense Matter”

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Is a color superconductor topological ?

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1. What is topological superconductor?
2. Is a color superconductor topological?
3. Possible physical implications





What is topological superconductor?

Rapid progress on discovery of new topological states of matter

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Examples of topological insulators

- quantum Hall effect (2D) found in 1980
- quantum spin Hall effect (2D) proposed in 2005 / found in 2007
- topological insulators (3D) proposed in 2007 / found in 2008

Examples of topological superconductors

- $p_x + ip_y$ superconductor (2D) possibly realized in Sr_2RuO_4
- B phase of ^3He (3D) based on the seminal paper by N. Read & D. Green in Phys. Rev. B (2000)
- ...

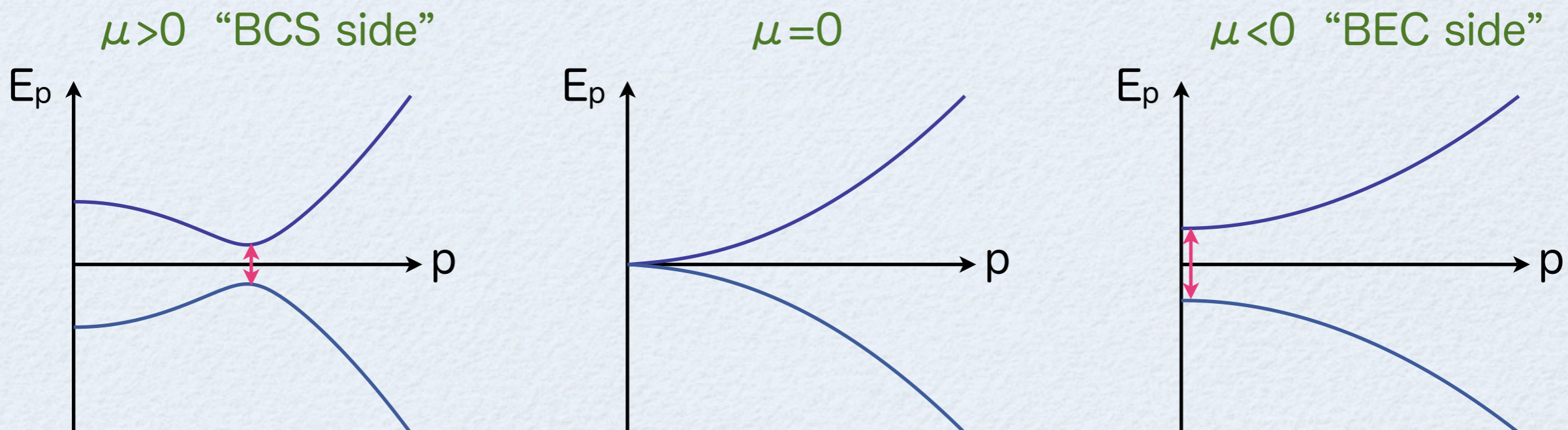
$p_x + ip_y$ superconductor in 2D

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- order parameter $\langle \psi_{-\mathbf{p}} \psi_{\mathbf{p}} \rangle = (p_x + ip_y) \Delta$
- mean-field Hamiltonian (in momentum space):

$$\mathcal{H}_{\mathbf{p}} = \begin{pmatrix} \frac{\mathbf{p}^2}{2m} - \mu & (p_x + ip_y) \Delta \\ (p_x - ip_y) \Delta & -\frac{\mathbf{p}^2}{2m} + \mu \end{pmatrix}$$
$$= \vec{h}_{\mathbf{p}} \cdot \vec{\sigma} \quad \text{with} \quad \vec{h}_{\mathbf{p}} = \left(p_x \Delta, -p_y \Delta, \frac{\mathbf{p}^2}{2m} - \mu \right)$$

- spectrum is gapped for $\mu \neq 0$: $E_{\mathbf{p}} = |\vec{h}_{\mathbf{p}}| = \sqrt{\left(\frac{\mathbf{p}^2}{2m} - \mu\right)^2 + \mathbf{p}^2 \Delta^2}$

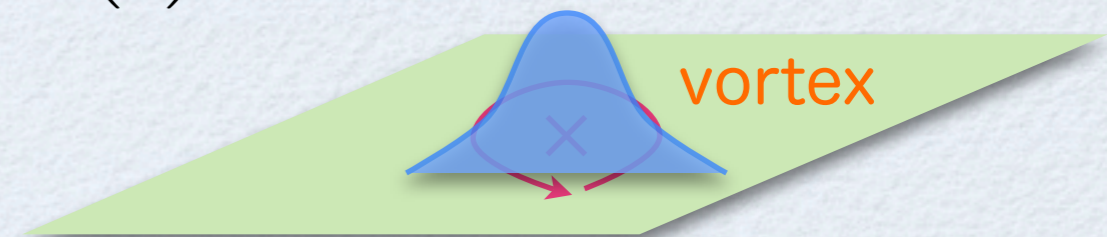


Bogoliubov-de Gennes equation

in the presence of a **vortex** : $\Delta(\mathbf{x}) = e^{i\theta} |\Delta(r)|$

$$\begin{pmatrix} \frac{\hat{\mathbf{p}}^2}{2m} - \mu & \frac{1}{2} \{ \hat{p}_-, \Delta(\mathbf{x}) \} \\ \frac{1}{2} \{ \hat{p}_+, \Delta^*(\mathbf{x}) \} & -\frac{\hat{\mathbf{p}}^2}{2m} + \mu \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = E \begin{pmatrix} u \\ v \end{pmatrix}$$

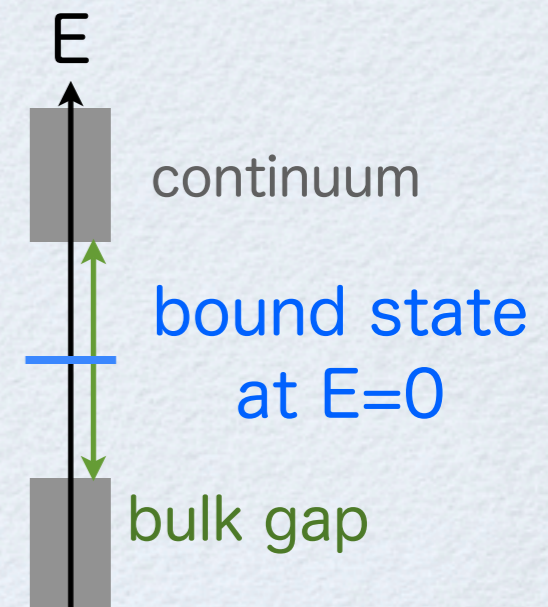
with $\hat{p}_{\pm} = -i\partial_x \pm \partial_y$



➡ **localized zero energy** solution but **only for $\mu > 0$**
(When $\mu < 0$, the solution exponentially grows)

$$\begin{pmatrix} u \\ v \end{pmatrix}_{r \rightarrow \infty} \rightarrow e^{i\frac{\pi}{4}} \begin{pmatrix} 1 \\ -i \end{pmatrix} J_0(\sqrt{2m\mu - (m|\Delta|)^2} r) e^{-m|\Delta|r}$$

Vortex in $p_x + ip_y$ superconductor for $\mu > 0$
supports a **localized gapless fermion**



Gapless edge state

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Bogoliubov-de Gennes equation
in the presence of a **boundary**

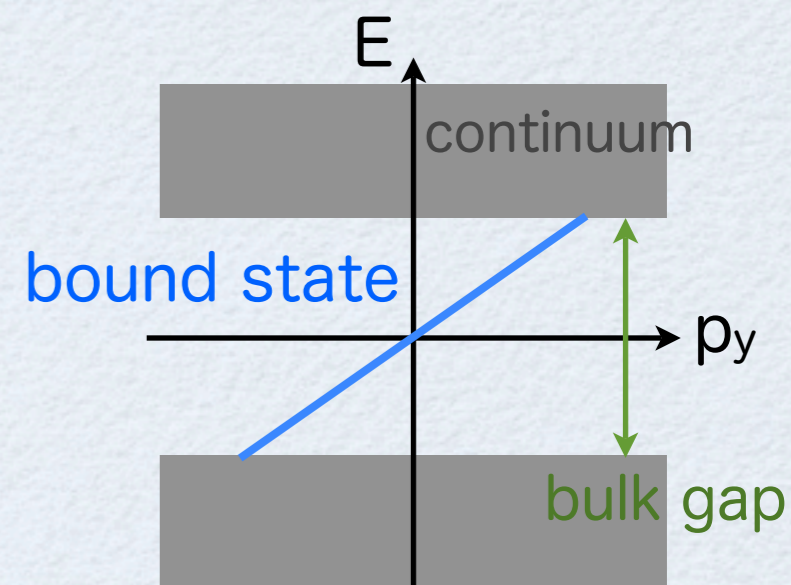
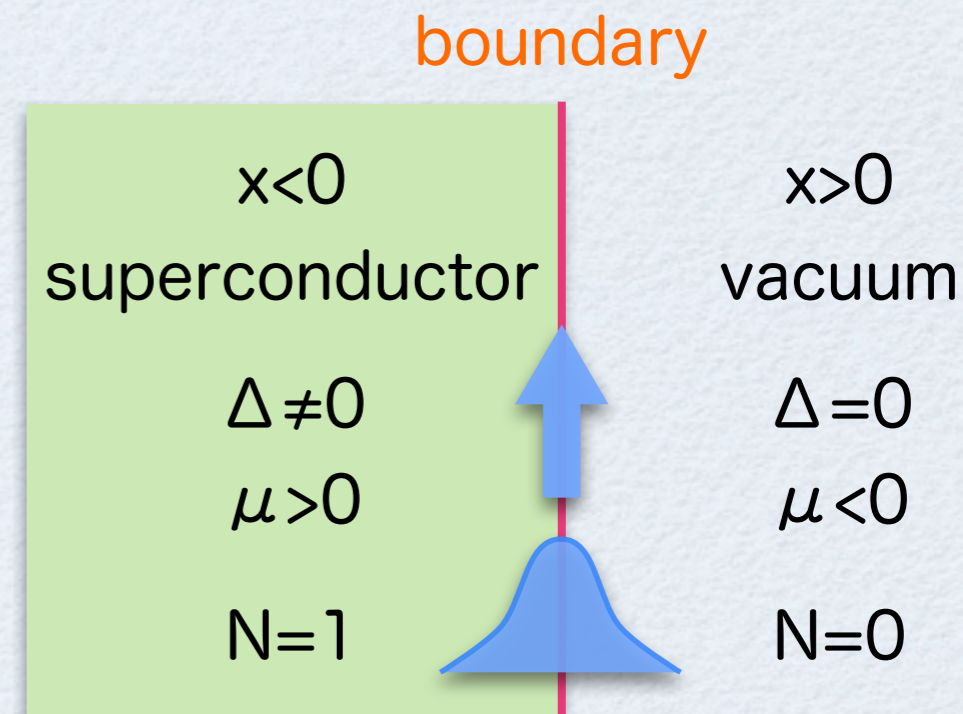
$$\begin{pmatrix} \frac{\hat{p}^2}{2m} - \mu & \hat{p}_- \Delta \\ \hat{p}_+ \Delta & -\frac{\hat{p}^2}{2m} + \mu \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = E \begin{pmatrix} u \\ v \end{pmatrix}$$

➔ **localized solution only for $\mu > 0$**

(When $\mu < 0$, the solution exponentially grows)

$$\begin{pmatrix} u \\ v \end{pmatrix}_{x < 0} = e^{-i\frac{\pi}{4}} \begin{pmatrix} 1 \\ i \end{pmatrix} e^{ip_y y + m\Delta x} \cos(\sqrt{2m\mu - (m\Delta)^2} x)$$

with linear dispersion : $E = \Delta p_y$



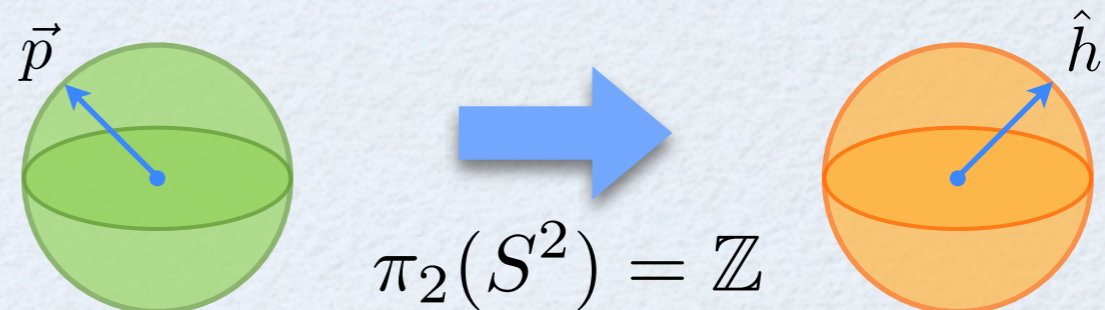
Boundary of $p_x + ip_y$ superconductor for $\mu > 0$
supports a **localized gapless fermion**

$$\mathcal{H}_{\mathbf{p}} = \begin{pmatrix} \frac{\mathbf{p}^2}{2m} - \mu & (p_x + ip_y) \Delta \\ (p_x - ip_y) \Delta & -\frac{\mathbf{p}^2}{2m} + \mu \end{pmatrix}$$

$$= \vec{h}_{\mathbf{p}} \cdot \vec{\sigma} \quad \text{with} \quad \vec{h}_{\mathbf{p}} = \left(p_x \Delta, -p_y \Delta, \frac{\mathbf{p}^2}{2m} - \mu \right)$$

Topological invariant can be defined for a gapped Hamiltonian in momentum space

$\hat{h}_{\mathbf{p}} \equiv \frac{\vec{h}_{\mathbf{p}}}{|\vec{h}_{\mathbf{p}}|}$ is a map from \mathbf{p}^2 -space ($\simeq S^2$) to S^2 (h-space)



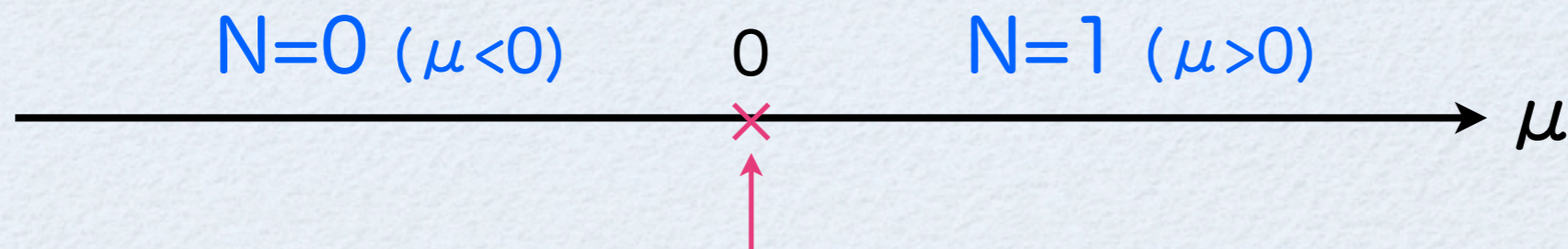
→ winding number $N = \frac{1}{8\pi} \int d\mathbf{p} \, \epsilon_{ab} \epsilon_{ijk} \hat{h}_i \partial_a \hat{h}_j \partial_b \hat{h}_k$

Topological phase transition

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$$N = \begin{cases} 1 & \text{for } \mu > 0 \\ 0 & \text{for } \mu < 0 \end{cases}$$

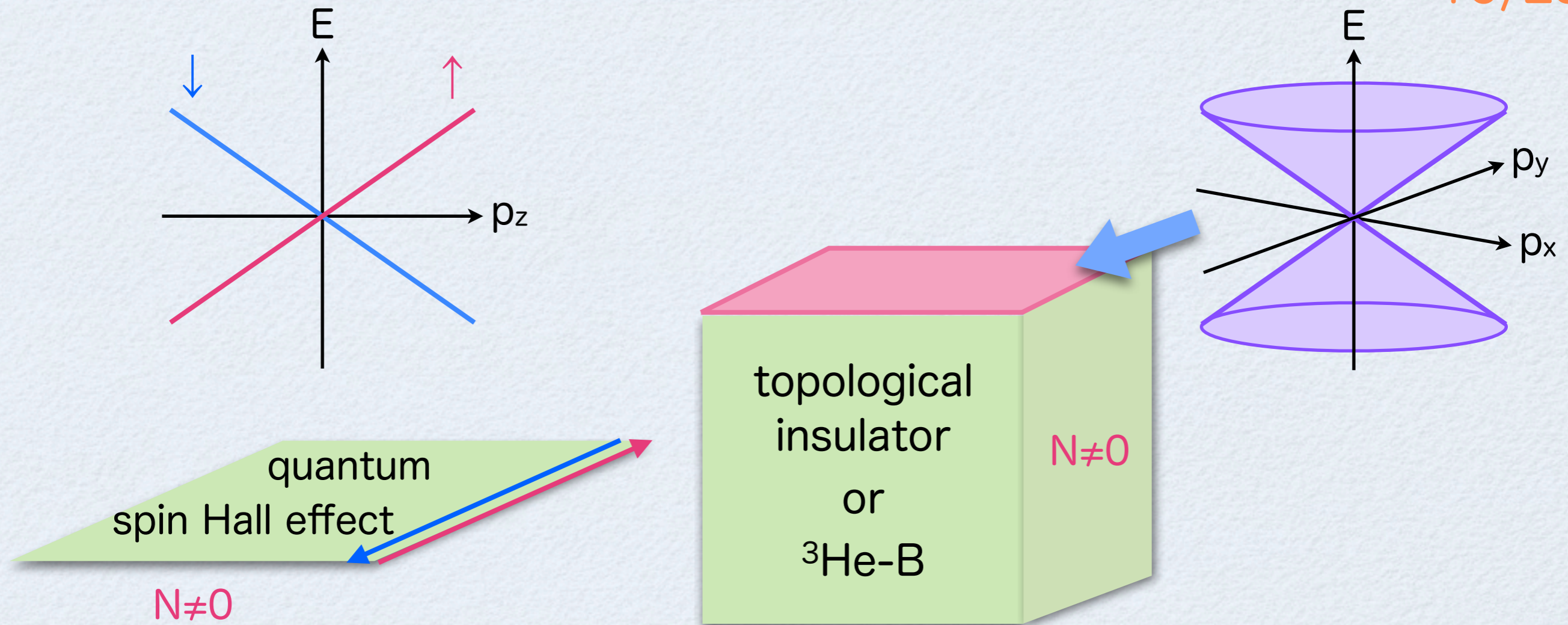
→ system is topological ! ($N \neq 0$)
→ system is not topological ($N=0$)



topological phase transition

(2 phases cannot be distinguished by symmetries)

1. Topological charge “N” can be defined for a **gapped Hamiltonian** in **momentum space**
2. Topological charge is invariant as long as the gap is open
3. The system is **topological** if $N \neq 0$
4. A boundary (vortex) supports a **localized gapless fermion**



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Classification of gapped states by topology

11/28

A. P. Schnyder et al., Phys. Rev. B (2008) & A. Kitaev, arXiv:0901.2686

Whether the topological charge (\mathbb{Z} or \mathbb{Z}_2) can be defined is determined by the symmetry and spatial dimension

name	symmetry			dimension			
	TRS	PHS	SLS	$d = 1$	$d = 2$	$d = 3$	
A	0	0	0	—	\mathbb{Z}	—	QHE
AI	+1	0	0	—	—	—	QSHE
AII	-1	0	0	—	\mathbb{Z}_2	\mathbb{Z}_2	
AIII	0	0	1	\mathbb{Z}	—	\mathbb{Z}	topological insulator
BDI	+1	+1	1	\mathbb{Z}	—	—	$p_x + ip_y$ SC
CII	-1	-1	1	\mathbb{Z}	—	\mathbb{Z}_2	
D	0	+1	0	\mathbb{Z}_2	\mathbb{Z}	—	$^3\text{He-B}$
C	0	-1	0	—	\mathbb{Z}	—	
DIII	-1	+1	1	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	
CI	+1	-1	1	—	—	\mathbb{Z}	

$$\mathcal{T}^{-1}\mathcal{H}\mathcal{T} = \mathcal{H}^*$$

$$\mathcal{T}^2 = \pm 1$$

$$\mathcal{C}^{-1}\mathcal{H}\mathcal{C} = -\mathcal{H}^*$$

$$\mathcal{C}^2 = \pm 1$$

$$\chi^{-1}\mathcal{H}\chi = -\mathcal{H}$$

Classification of gapped states by topology

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DIII	-1	+1	1	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	$^3\text{He-B}$
CI	+1	-1	1	—	—	\mathbb{Z}	

These classifications are for noninteracting Hamiltonians
(open problem for interacting Hamiltonians)

Classification of gapped states by topology

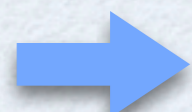
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A. P. Schnyder et al., Phys. Rev. B (2008) & A. Kitaev, arXiv:0901.2686

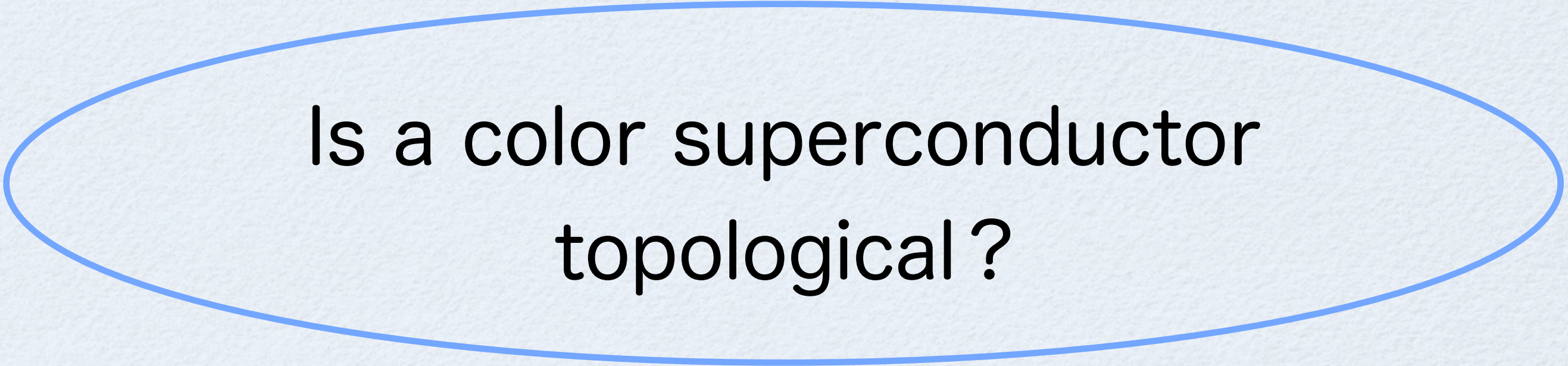
Any (noninteracting) gapped states of matter can be asked if it is topological or not

	TRS	PHS	SLS	$d = 1$	$d = 2$	$d = 3$
A	0	0	0	—	\mathbb{Z}	—
AI	+1	0	0	—	—	—
AII	−1	0	0	—	\mathbb{Z}_2	\mathbb{Z}_2
AIII	0	0	1	\mathbb{Z}	—	\mathbb{Z}
BDI	+1	+1	1	\mathbb{Z}	—	—
CII	−1	−1	1	\mathbb{Z}	—	\mathbb{Z}_2
D	0	+1	0	\mathbb{Z}_2	\mathbb{Z}	—
C	0	−1	0	—	\mathbb{Z}	—
DIII	−1	+1	1	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
CI	+1	−1	1	—	—	\mathbb{Z}

color
superconductor



Is a color superconductor topological ?



Is a color superconductor
topological?

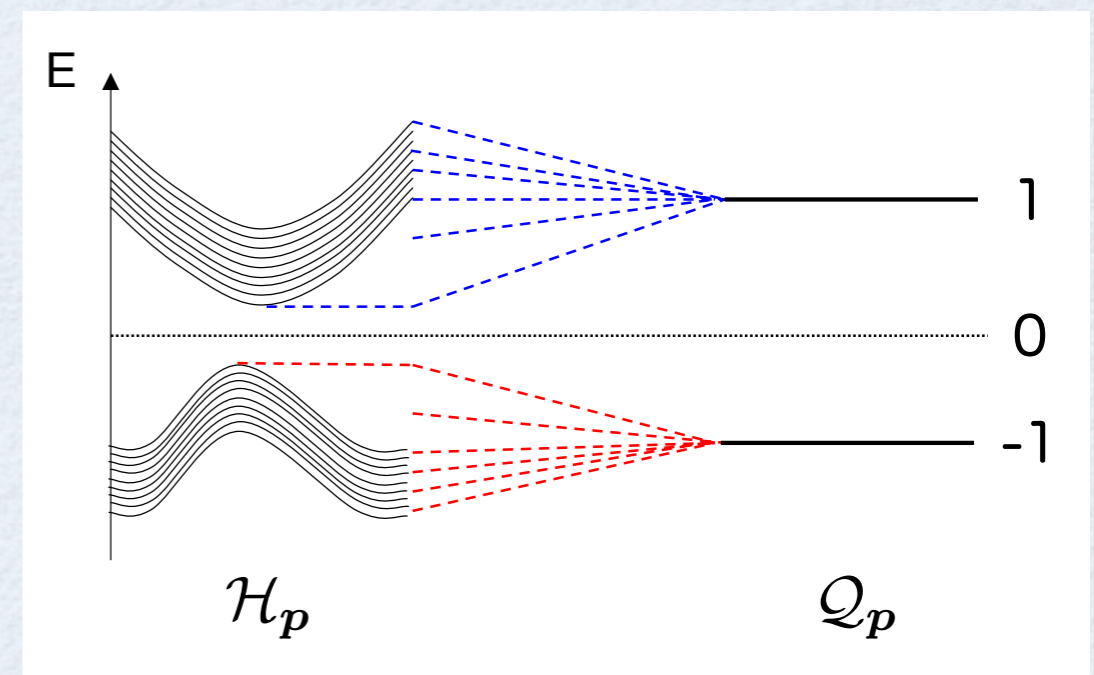
Topological charge for “class DIII” Hamiltonians in 3D

$$(\mathcal{C}^{-1}\mathcal{H}\mathcal{C} = -\mathcal{H}^* \quad \mathcal{T}^{-1}\mathcal{H}\mathcal{T} = \mathcal{H}^* \quad \text{with} \quad \mathcal{C}^2 = 1 \quad \mathcal{T}^2 = -1)$$

Hamiltonian in momentum space

$$\mathcal{H}_{\mathbf{p}} = U_{\mathbf{p}} \begin{pmatrix} \mathcal{E}_{\mathbf{p}} & 0 \\ 0 & -\mathcal{E}_{\mathbf{p}} \end{pmatrix} U_{\mathbf{p}}^{\dagger}$$

$$\mathcal{Q}_{\mathbf{p}} \equiv U_{\mathbf{p}} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} U_{\mathbf{p}}^{\dagger} \rightarrow \begin{pmatrix} 0 & q_{\mathbf{p}} \\ q_{\mathbf{p}}^{\dagger} & 0 \end{pmatrix}$$



$q_{\mathbf{p}} \in \text{U}(n)$ is a map from \mathbf{p}^3 -space to unitary matrix

➡ winding number : $\pi_3[\text{U}(n)] = \mathbb{Z} \quad (n \geq 2)$

$$N = \frac{1}{24\pi^2} \int d\mathbf{p} \epsilon^{ijk} \text{Tr}[(q_{\mathbf{p}}^{-1} \partial_i q_{\mathbf{p}})(q_{\mathbf{p}}^{-1} \partial_j q_{\mathbf{p}})(q_{\mathbf{p}}^{-1} \partial_k q_{\mathbf{p}})]$$

- equal quark masses

$$m_u = m_d = m_s = m$$

- color-flavor-locked pairing

$$\langle \psi_{a,f}^T C \gamma^5 \psi_{b,g} \rangle = \Delta \epsilon_{Iab} \epsilon_{Ifg}$$

(parity even : $\Delta_R = \Delta_L$)



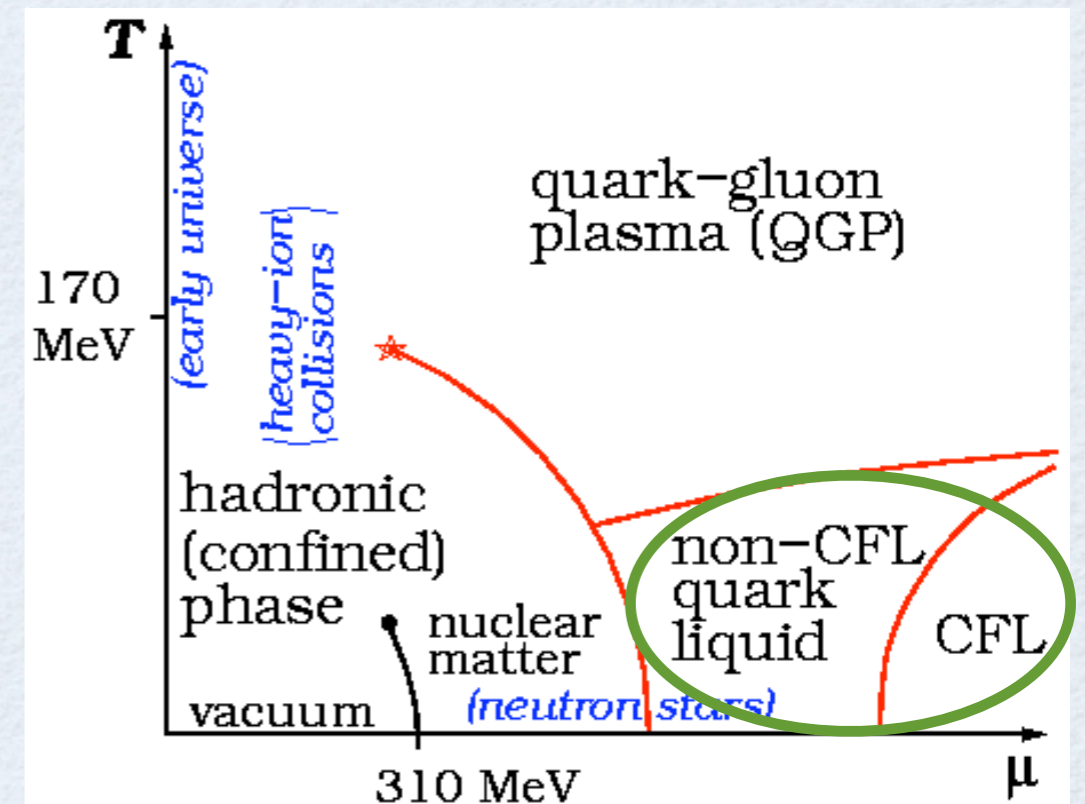
All 9 quarks are gapped

each sector is described by

$$\mathcal{H} = \begin{pmatrix} -i\boldsymbol{\alpha} \cdot \boldsymbol{\partial} + \beta m - \mu & \Delta(\mathbf{x}) \\ \Delta^*(\mathbf{x}) & i\boldsymbol{\alpha} \cdot \boldsymbol{\partial} - \beta m + \mu \end{pmatrix} \otimes 9$$

For comparison, we also consider parity-odd pairing

$$\langle \psi_{a,f}^T C \psi_{b,g} \rangle = \Delta \epsilon_{Iab} \epsilon_{Ifg} \quad \Leftrightarrow \quad \Delta_R = -\Delta_L$$



Topological charge

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free-space Hamiltonian with $\Delta(\mathbf{x}) = \Delta_0$: constant

In the chiral limit ($m=0$) : $\mathcal{H} = \mathcal{H}_R + \mathcal{H}_L \Rightarrow N = N_R + N_L$

even parity pairing

$$N_R = \frac{\Delta_0}{2|\Delta_0|} \quad N_L = -\frac{\Delta_0}{2|\Delta_0|}$$

topologically nontrivial



$m \neq 0$

$$N = 0$$

topologically trivial

odd parity pairing

$$N_R = \frac{\Delta_0}{2|\Delta_0|} \quad N_L = \frac{\Delta_0}{2|\Delta_0|}$$



$m \neq 0$

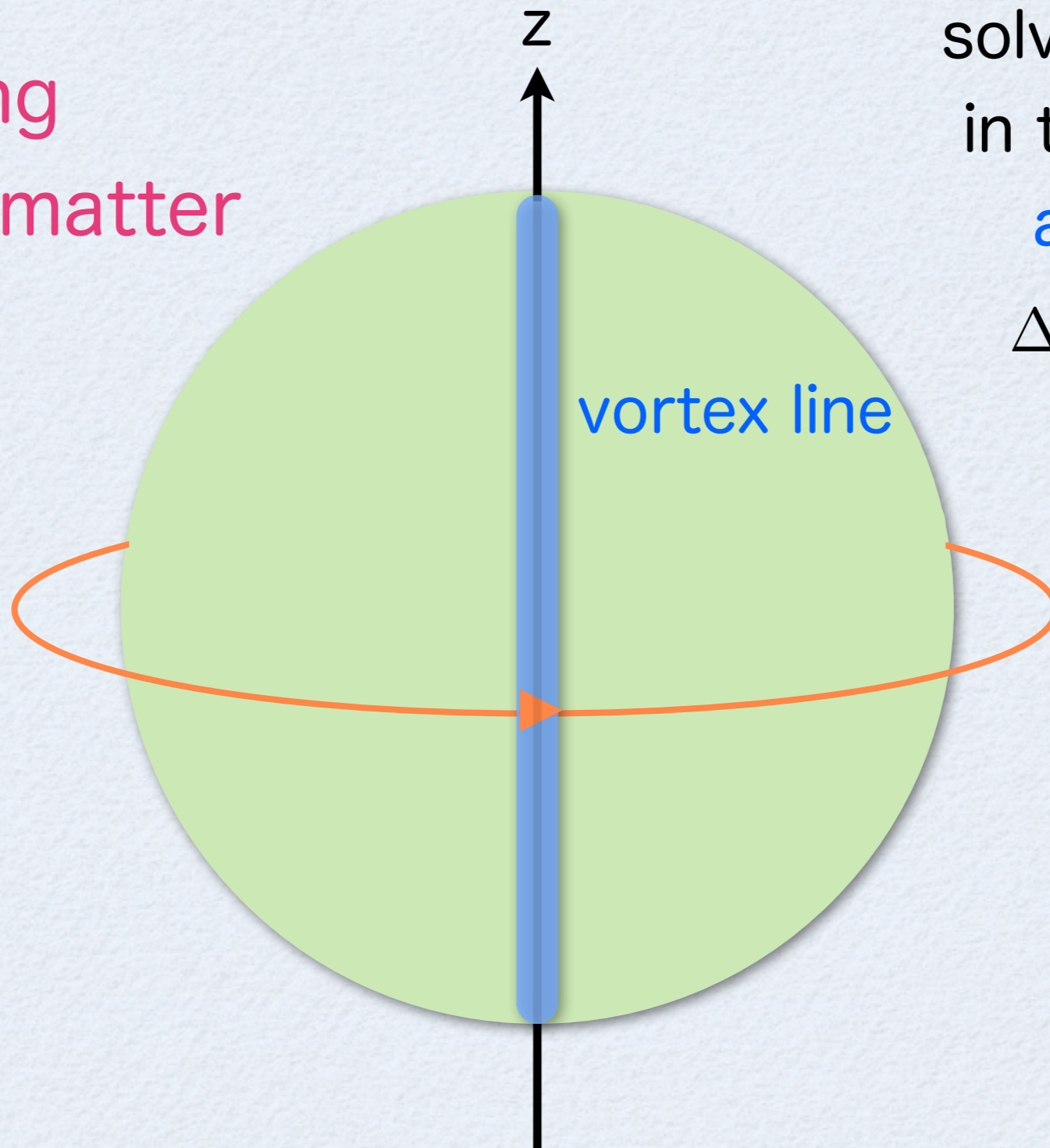
$$N = \frac{\Delta_0}{|\Delta_0|} \quad \text{for } m^2 < \mu^2 + \Delta_0^2$$



topological
phase transition

$$N = 0 \quad \text{for } m^2 > \mu^2 + \Delta_0^2$$

rotating
CFL quark matter



solve BdG equation
in the presence of
a vortex line :

$$\Delta(\boldsymbol{x}) = e^{i\theta} |\Delta(r)|$$

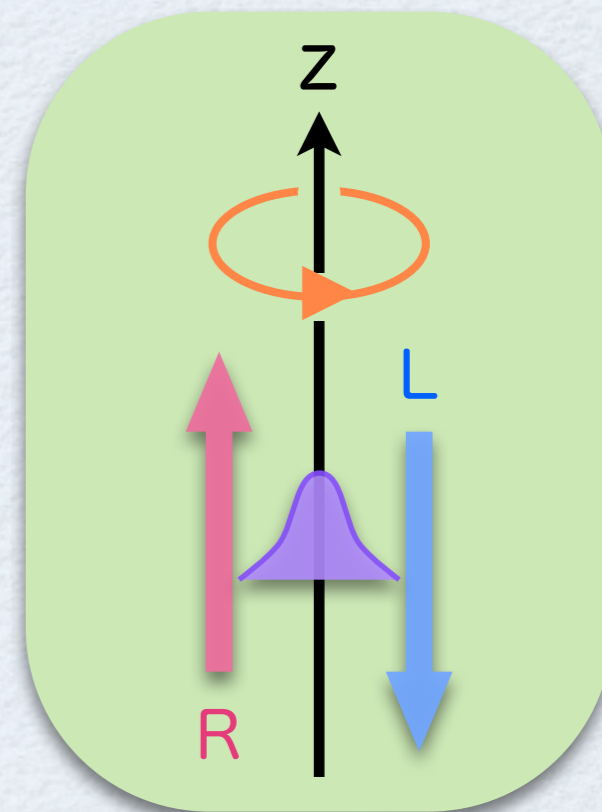
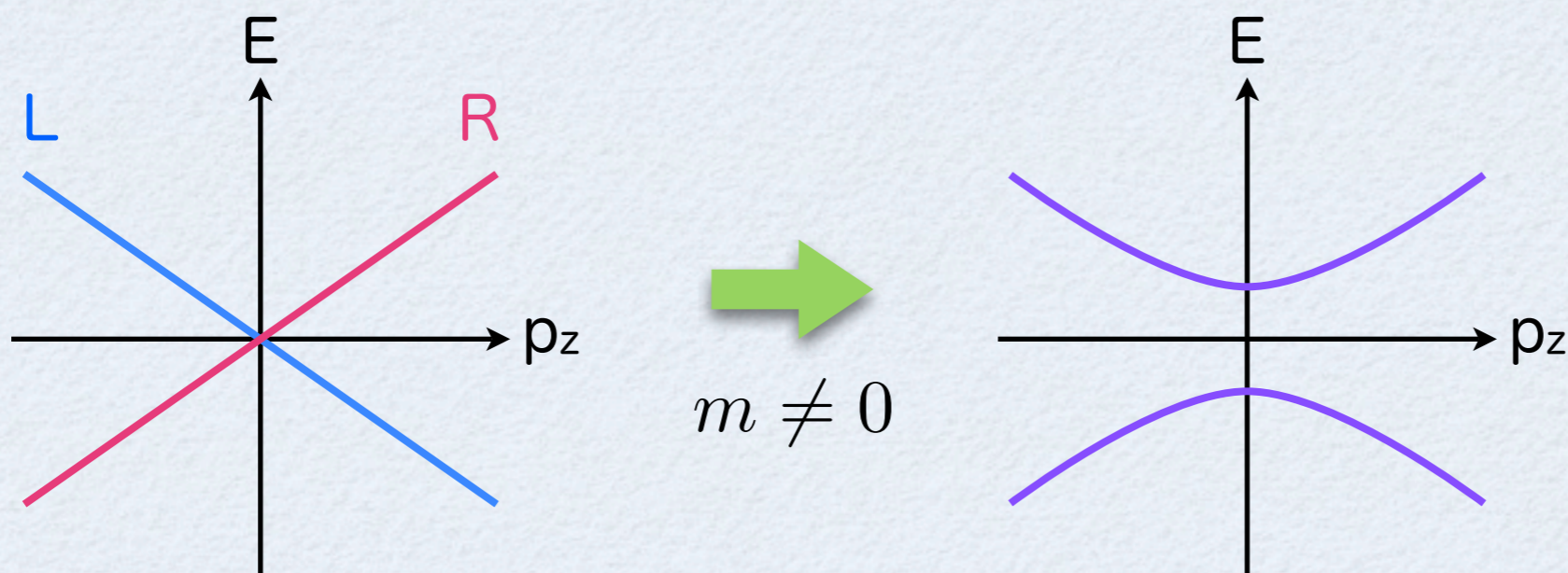
Fermion spectrum @ vortex

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even parity pairing

$m=0$ ($N_R, N_L \neq 0$) : 2 localized gapless fermions

- $E = vp_z$ for right-handed
- $E = -vp_z$ for left-handed



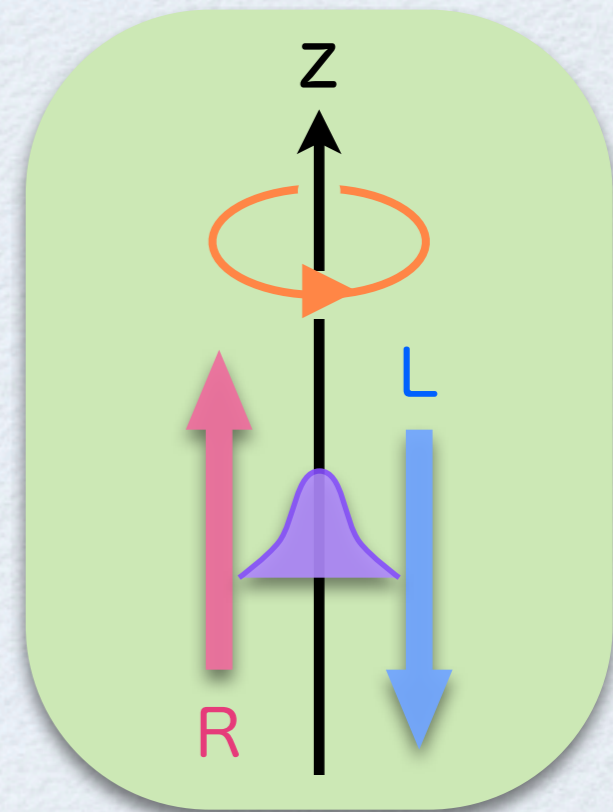
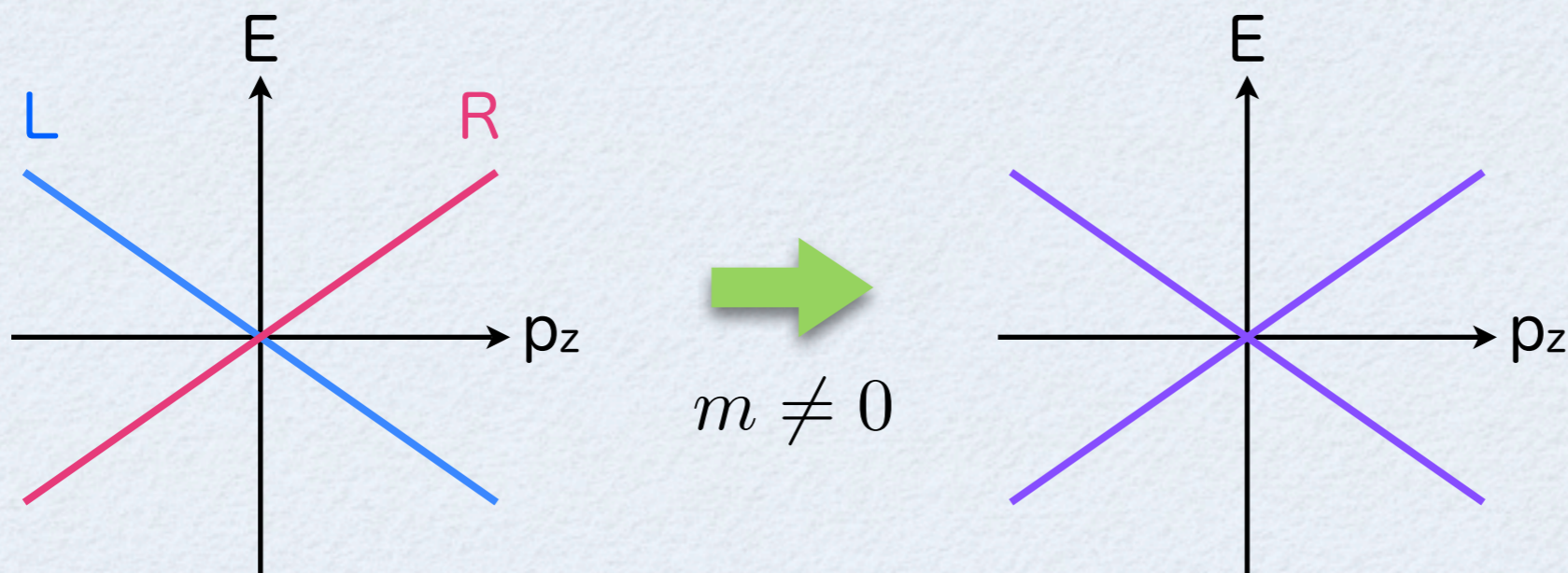
$m \neq 0$ ($N=0$) : 2 localized fermions become gapped

$$E = \pm v \sqrt{m^2 + p_z^2}$$

odd parity pairing

$m=0$ ($N_R, N_L \neq 0$) : 2 localized gapless fermions

- $E = vp_z$ for right-handed
- $E = -vp_z$ for left-handed



- $m \neq 0$
- $m^2 < \mu^2 + |\Delta(\infty)|^2$ ($N \neq 0$) : gaplessness is preserved
 - $m^2 > \mu^2 + |\Delta(\infty)|^2$ ($N = 0$) : localized fermions disappear

Effective 1D theory @ vortex

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Bulk fermions are gapped by $\Delta(r \rightarrow \infty)$

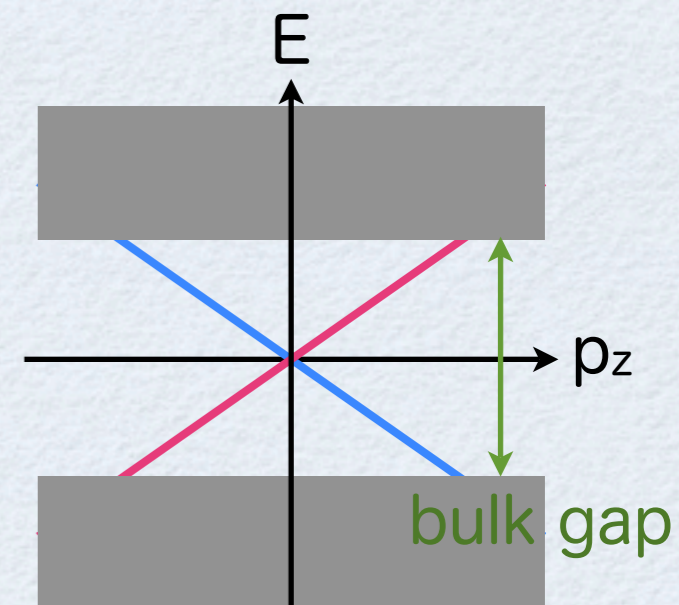
Low-energy effective theory should involve (nearly) gapless fermions on a vortex line

$$H_{1D} = \frac{v}{2} \int \frac{dp_z}{2\pi} (p_z \psi_{p_z}^{R\dagger} \psi_{p_z}^R - p_z \psi_{p_z}^{L\dagger} \psi_{p_z}^L)$$

velocity

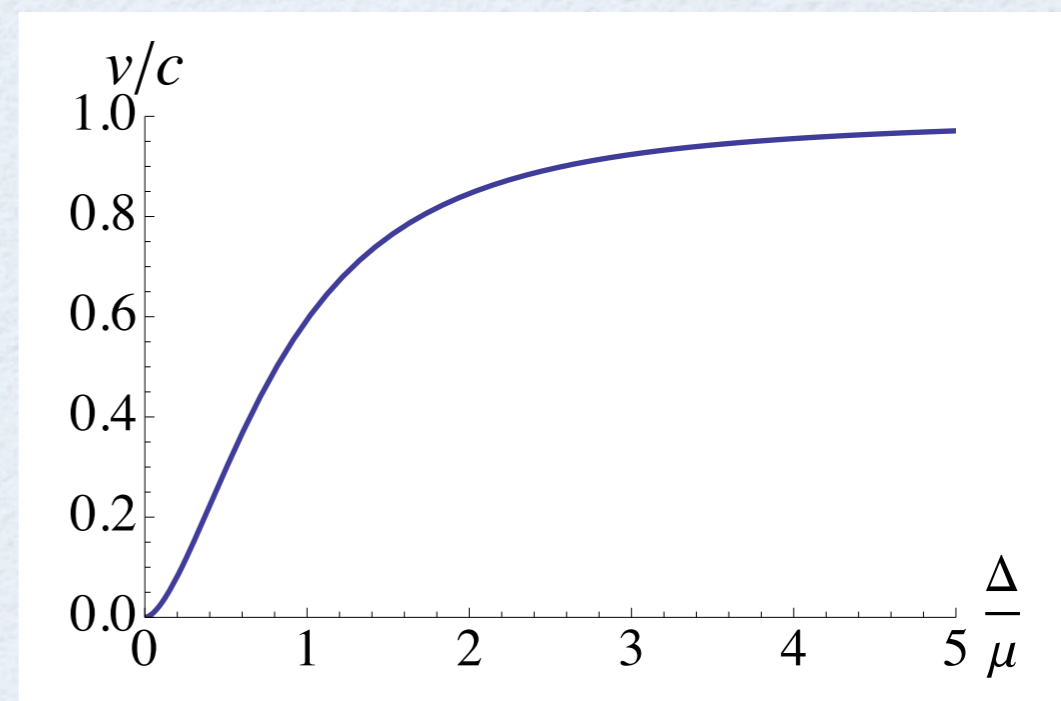
$$v = \frac{\int_0^\infty dr r [J_0^2(\mu r) - J_1^2(\mu r)] e^{-2 \int_0^r |\Delta(r')| dr'}}{\int_0^\infty dr r [J_0^2(\mu r) + J_1^2(\mu r)] e^{-2 \int_0^r |\Delta(r')| dr'}}$$

simple case
 $\Delta(r) = \Delta$



Majorana fermions

$$\psi_{p_z}^{R(L)\dagger} = \psi_{-p_z}^{R(L)}$$



Effective 1D theory @ vortex

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Bulk fermions are gapped by $\Delta(r \rightarrow \infty)$

Low-energy effective theory should involve (nearly) gapless fermions on a vortex line

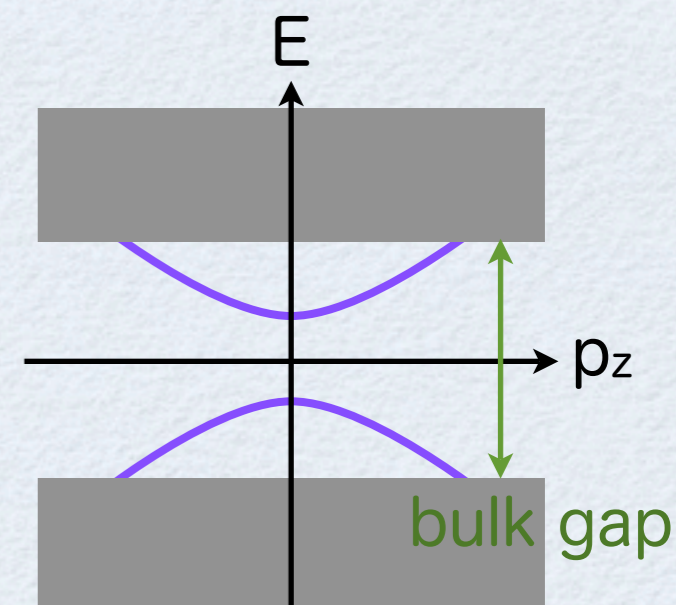
$$H_{1D} = \frac{v}{2} \int \frac{dp_z}{2\pi} \left(p_z \psi_{p_z}^{R\dagger} \psi_{p_z}^R - p_z \psi_{p_z}^{L\dagger} \psi_{p_z}^L \right. \\ \left. + im \psi_{p_z}^{R\dagger} \psi_{p_z}^L - im \psi_{p_z}^{L\dagger} \psi_{p_z}^R \right)$$

velocity

mass term for even parity pairing

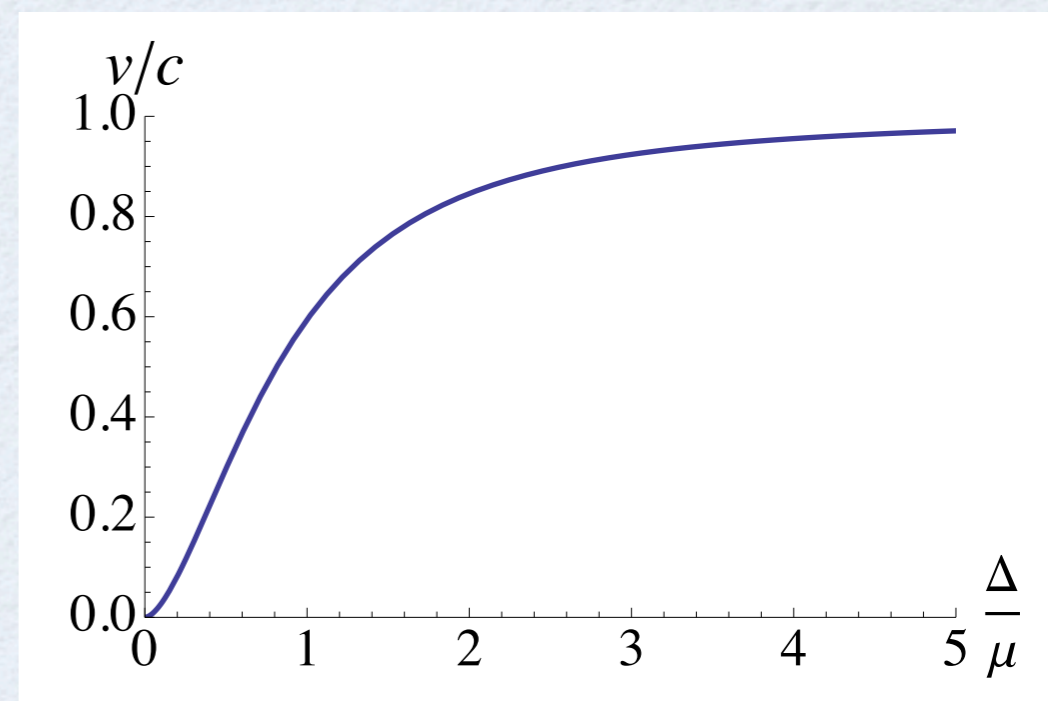
$$v = \frac{\int_0^\infty dr r [J_0^2(\mu r) - J_1^2(\mu r)] e^{-2 \int_0^r |\Delta(r')| dr'}}{\int_0^\infty dr r [J_0^2(\mu r) + J_1^2(\mu r)] e^{-2 \int_0^r |\Delta(r')| dr'}}$$

simple case
 $\Delta(r) = \Delta$



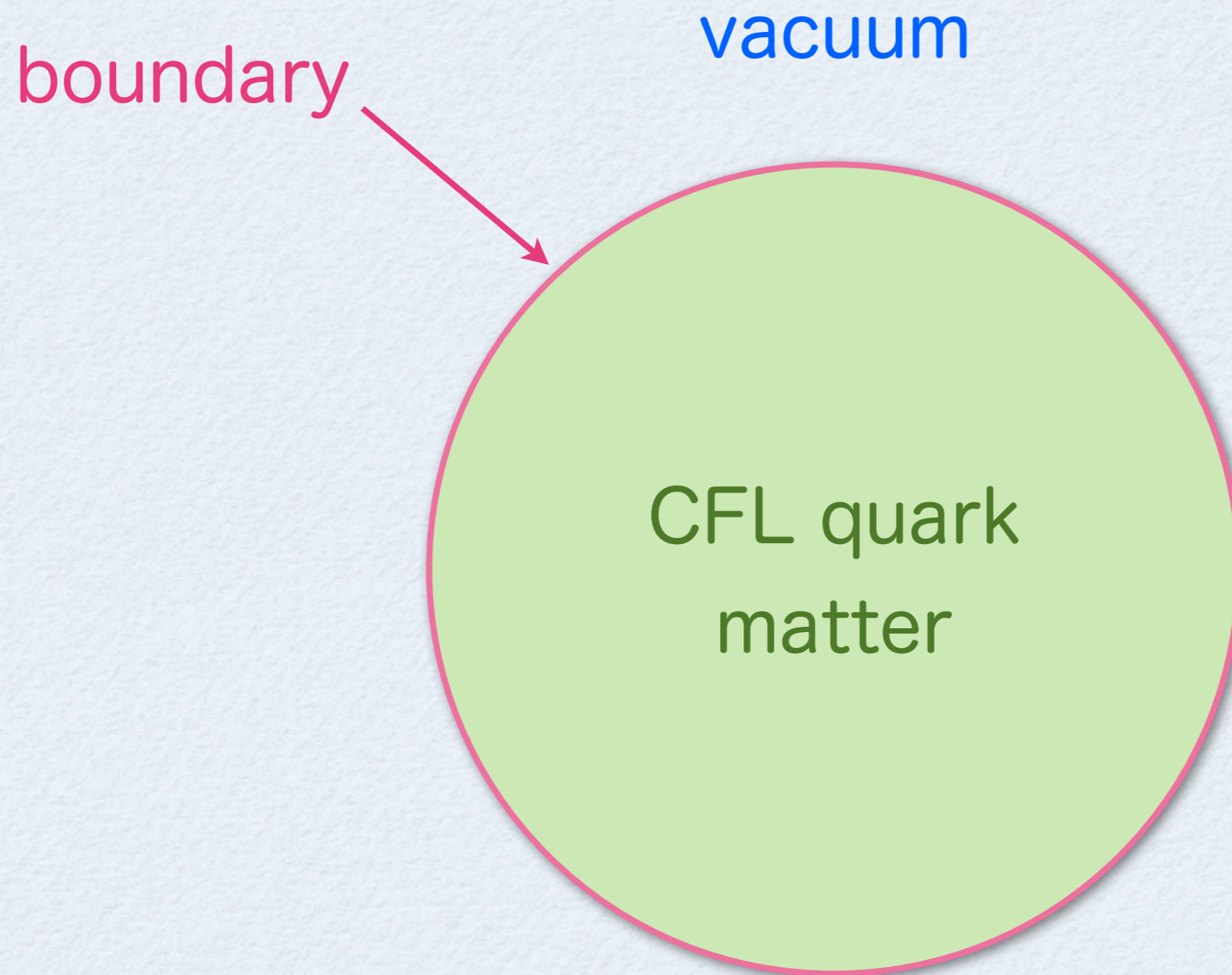
Majorana fermions

$$\psi_{p_z}^{R(L)\dagger} = \psi_{-p_z}^{R(L)}$$



gapless fermion @ boundary ?

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gapless fermion @ boundary ?

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odd parity pairing

$z > 0$

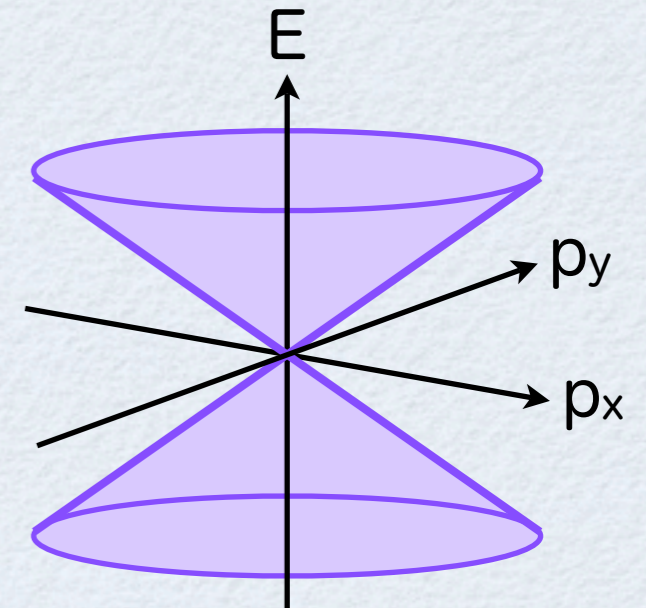
chiral-broken
vacuum

$\mu, \Delta = 0$

$m > 0$

$N=0$

$$E = \pm v \sqrt{p_x^2 + p_y^2}$$



localized gapless fermion

$$H_{2D} = \frac{v}{2} \int \frac{d\mathbf{p}_\perp}{(2\pi)^2} \psi_{\mathbf{p}_\perp}^\dagger (\boldsymbol{\sigma}_\perp \cdot \mathbf{p}_\perp) \psi_{\mathbf{p}_\perp}$$

$$\text{with } v = \left| 1 + \frac{m}{\Delta + i\mu} \right| / \left(1 + \frac{m}{\Delta} \right)$$

$$\text{Majorana fermion : } \psi_{\mathbf{p}_\perp}^\dagger = \psi_{-\mathbf{p}_\perp}^T \sigma_1$$

$z < 0$

color
superconductor

$\mu, \Delta > 0$

$m=0$

$N=1$

gapless fermion @ boundary ?

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even parity pairing

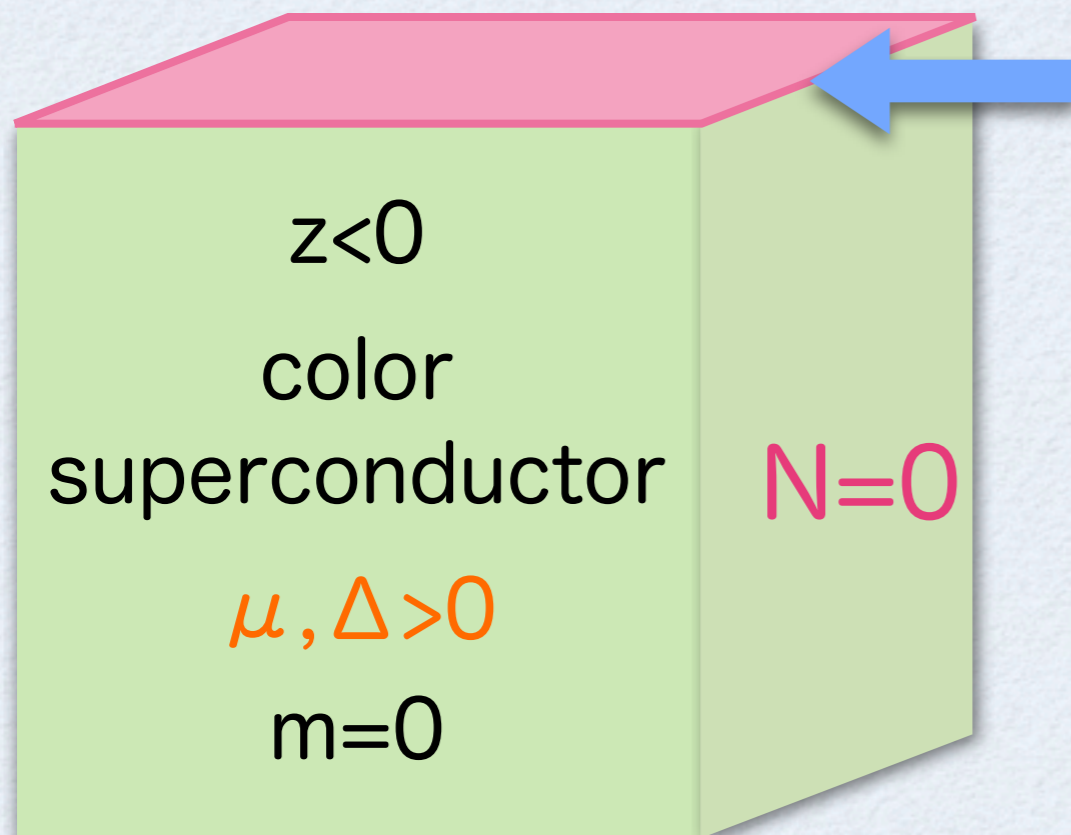
$z > 0$

chiral-broken
vacuum

$\mu, \Delta = 0$

$m > 0$

$N=0$



No gapless fermion



Conclusions

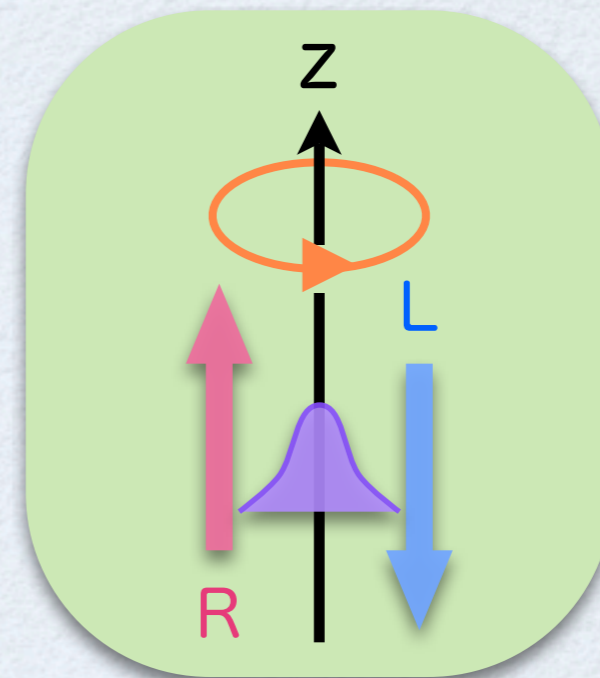
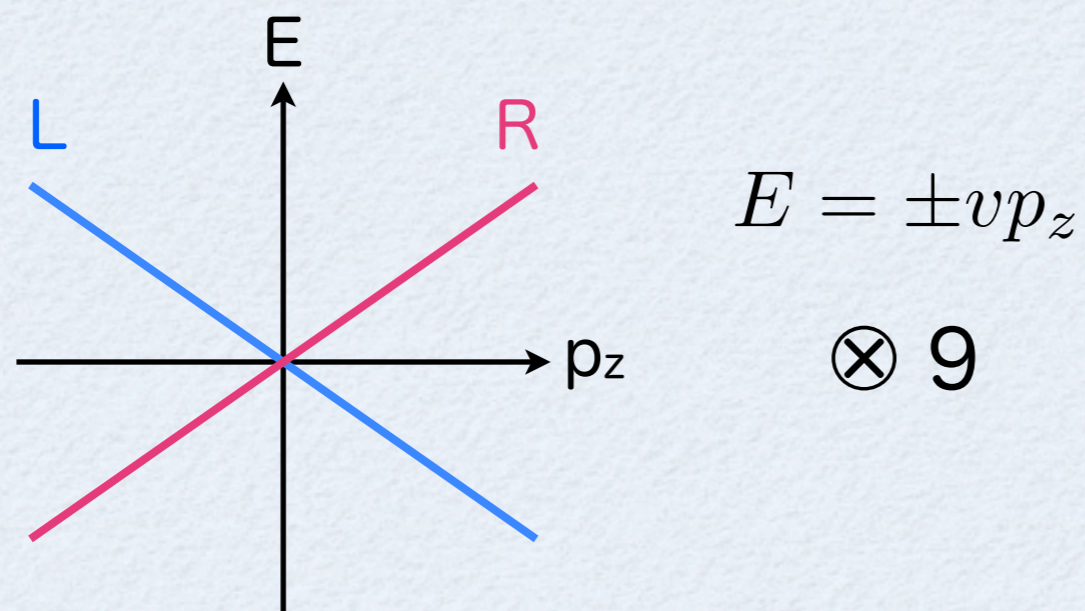
Is a color superconductor topological ?

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Yes, in the chiral limit ($N_R, N_L \neq 0$)

➔ $U_B(1)$ vortex in rotating CFL quark matter supports 9 sets of gapless right- and left-handed quarks

5 on a non-Abelian vortex [Yasui, Itakura, Nitta (arXiv:1001.3730)]



➔ Microscopic origin of axial current flowing on a vortex derived using anomalies

$$J_z^5 = \frac{\mu}{2\pi} \times (\text{vorticity})$$

D. T. Son & A. R. Zhitnisky, Phys. Rev. D (2004)

Is a color superconductor topological ?

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No, with nonzero quark mass ($N=0$)

→ localized quarks become gapped

$$E = \pm v \sqrt{m^2 + p_z^2}$$

- much lighter than bulk quarks

$$vm/\Delta \sim 5 \times 10^{-3} \quad \text{for } \mu \sim 500 \text{ MeV} \quad \Delta \sim 50 \text{ MeV} \quad m \sim 10 \text{ MeV}$$

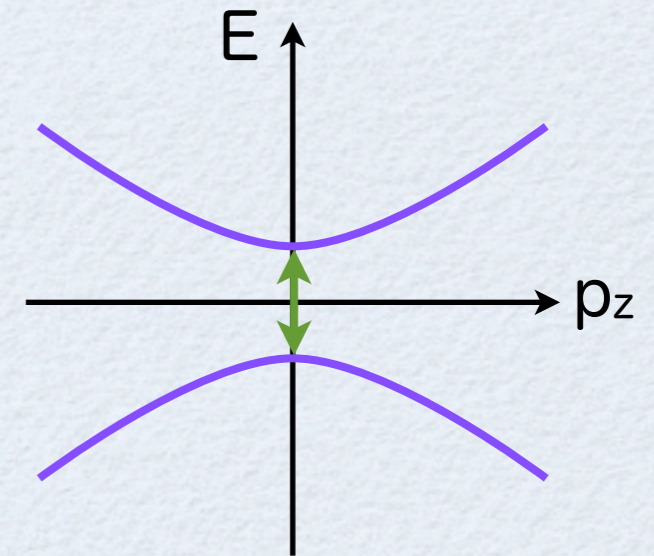
- parametrically lighter than pseudo-NG bosons

$$vm \rightarrow m (\Delta/\mu)^2 \ln(\mu/\Delta) \ll m_{\text{NG}} \rightarrow m (\Delta/\mu) \quad \text{D. T. Son \& M. A. Stephanov} \\ \text{Phys. Rev. D (2000)}$$

→ could be important low-energy degrees of freedom

- impact on the physics of rotating neutron/quark stars?
(transport property, neutrino emissivity, pulsar kick, ...)

- topology of Hamiltonian \Leftrightarrow existence of gapless fermions
- effect of interaction







Backup slides

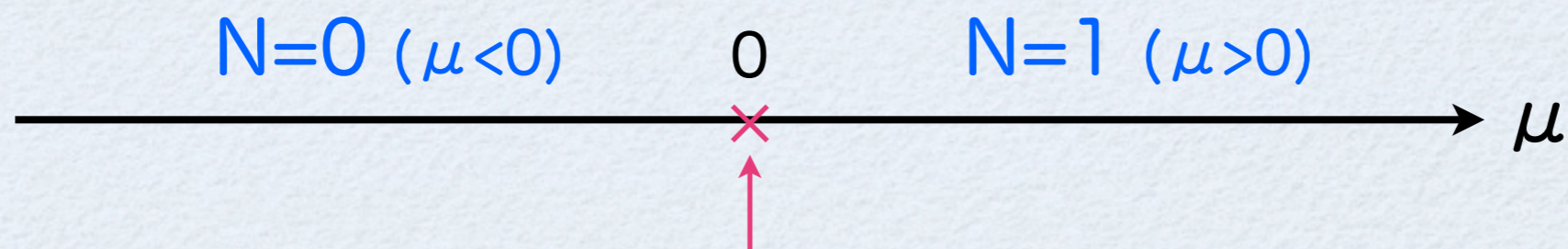
Topological phase transition

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$$N = \begin{cases} 1 & \text{for } \mu > 0 \\ 0 & \text{for } \mu < 0 \end{cases}$$

 system is topological ! ($N \neq 0$)

 system is not topological ($N=0$)



topological phase transition

(2 phases cannot be distinguished by symmetries)

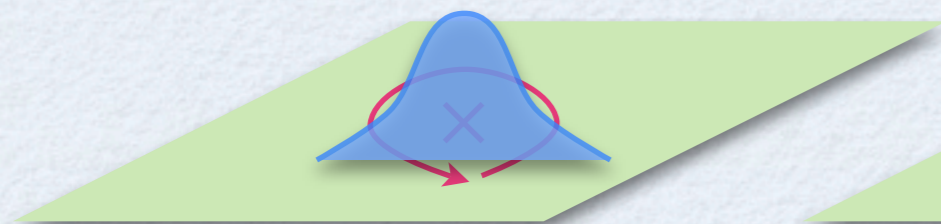
- Topological charge is invariant under the smooth deformation of Hamiltonian
- Topological charge can change only when the gap in spectrum closes

 What are physical consequences of nontrivial topology?

We have learned ...

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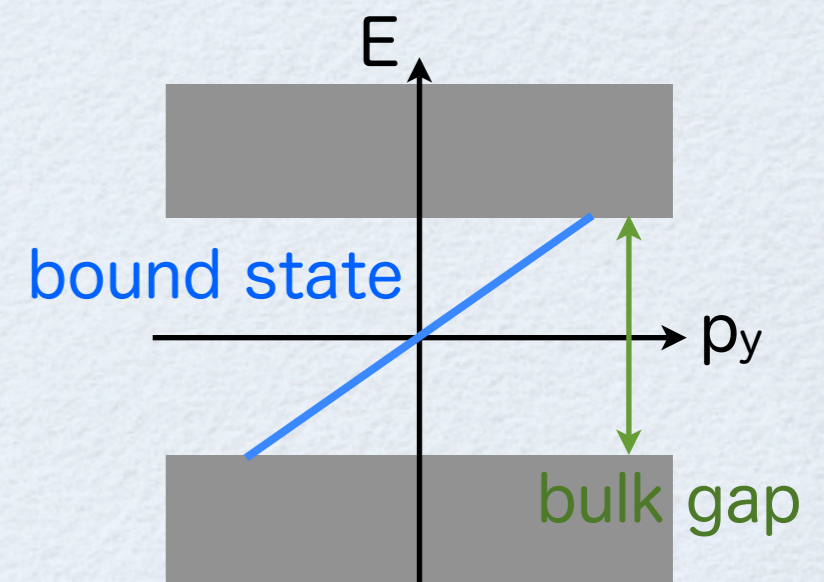
1. Topological charge “N” can be defined for a **gapped Hamiltonian** in **momentum space**
 2. Topological charge is invariant as long as the gap is open
 3. The system is **topological** if $N \neq 0$
- ↓
4. A vortex / boundary supports a **localized gapless fermion**
 5. $p_x + ip_y$ superconductor in 2D for $\mu > 0$ is one such example



vortex



boundary



Space of gapped Hamiltonians

